Chapter 1: The Reals

Cummings' Exercises

Exercises with a $\star$ are considered "Notable Exercises" in the book.

Exercise 1.8. $\star$

(a) Prove that $(A \cap B)^c = A^c \cup B^c$
(b) Prove that $(A \cup B)^c = A^c \cap B^c$

Exercise 1.13. $\star$

Let a, b and ε be elements of an ordered field.

(a) Show that if $a < b + \varepsilon$ for every $\varepsilon > 0$, then $a \leq b$.
(b) Use part (a) to show that if $|a - b| < \varepsilon$ for all $\varepsilon > 0$, then $a = b$.

Exercise 1.23. $\star$

Assume that $A \subseteq B$ and both are bounded above. Prove that $\sup(A) \leq \sup(B)$.

Exercise 1.30.

Let $A, B \subseteq \mathbb{R}$, and assume that $\sup(A) < \sup(B)$.

(a) Show that there exists an element $b \in B$ that is an upper bound for A.
(b) Give an example to show that this is not necessarily the case if we instead only assume that $\sup(A) \leq \sup(B)$. You do not need to prove your answer

Exercise 1.34. $\star$

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n]$. Also, assume that each $I_n + 1$ is contained inside I_n. This gives a sequence of increasingly smaller intervals,

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq ...$$

Prove that $\cap_{n=1}^{\infty} I_n \neq \emptyset$. That is, prove that there is some real number x such that $x \in I_n$ for every $n \in \mathbb{N}$.

MIT Exercises

From MIT OCW MATH 18.100A/18.1001 assignments 1 and 2, and recitation 1.

Exercise 3 from Homework 1

Show that for a finite set A of cardinality n, the cardinality of $\mathcal{P}(A)$ is 2^n.

Exercise 6 from Homework 1

In this exercise, you will prove that

$$|\{q \in \mathbb{Q} : q > 0\}| = |\mathbb{N}|.$$

In what follows, we will use the following theorem without proof:

Theorem. Let $q \in \mathbb{Q}$ with $q > 0$. Then

1) If $q \in \mathbb{N}$ and $q \neq 1$, then there exist unique prime numbers $p_1 < p_2 < \cdots < p_N$ and unique exponents $r_1, \ldots, r_N \in \mathbb{N}$ such that

$$q = p_1^{r_1} p_2^{r_2} \cdots p_N^{r_N}, \qquad (\dagger)$$

2) If $q \notin \mathbb{N}$, then there exist unique prime numbers $p_1 < p_2 < \cdots < p_N$, $q_1 < q_2 < \cdots < q_M$ with $p_i \neq q_j$ for all $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, M\}$, and unique exponents $r_1, \ldots, r_N, s_1, \ldots, s_M \in \mathbb{N}$ such that

$$q = \frac{p_1^{r_1} p_2^{r_2} \cdots p_N^{r_N}}{q_1^{s_1} q_2^{s_2} \cdots q_M^{s_M}}. \qquad (\ddagger)$$

Define $f : \{q \in \mathbb{Q} : q > 0\} \to \mathbb{N}$ as follows: $f(1) = 1$, if $q \in \mathbb{N} \setminus \{1\}$ is given by $(\dagger)$, then

$$f(q) = p_1^{2r_1} \cdots p_N^{2r_N},$$

and if $q \in \mathbb{Q} \setminus \mathbb{N}$ is given by $(\ddagger)$, then

$$f(q) = p_1^{2r_1} \cdots p_N^{2r_N} \cdot q_1^{2s_1 - 1} \cdots q_M^{2s_M - 1}.$$

(a) Compute $f(4/15)$. Find q such that $f(q) = 108$.

(b) Use the **Theorem** to prove that f is a bijection.

Exercise 1 from Homework 2

Let F be an ordered field and $x, y, z \in F$. Prove that if $x < 0$ and $y < z$, then $xy > xz$.

Exercise 2 from Homework 2

Let S be an ordered set. Let $A \subset S$ be a nonempty finite subset. Show that A is bounded. Furthermore, show that $\inf(A)$ exists and is in A, and $\sup(A)$ exists and is in A. (Hint: Use induction)